

# POSITIVE AND SIGN CHANGING SOLUTIONS TO A NONLINEAR CHOQUARD EQUATION

MÓNICA CLAPP AND DORA SALAZAR

ABSTRACT. We consider the problem

$$-\Delta u + W(x)u = \left( \frac{1}{|x|^\alpha} * |u|^p \right) |u|^{p-2}u, \quad u \in H_0^1(\Omega),$$

where  $\Omega$  is an exterior domain in  $\mathbb{R}^N$ ,  $N \geq 3$ ,  $\alpha \in (0, N)$ ,  $p \in [2, \frac{2N-\alpha}{N-2})$ ,  $W \in C^0(\mathbb{R}^N)$ ,  $\inf_{\mathbb{R}^N} W > 0$ , and  $W(x) \rightarrow V_\infty > 0$  as  $|x| \rightarrow \infty$ . Under symmetry assumptions on  $\Omega$  and  $W$ , which allow finite symmetries, and some assumptions on the decay of  $W$  at infinity, we establish the existence of a positive solution and multiple sign changing solutions to this problem, having small energy.

KEY WORDS: Nonlinear Choquard equation; nonlocal nonlinearity; exterior domain; positive and sign changing solutions.

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## 1. INTRODUCTION

We consider the problem

$$(1.1) \quad \begin{cases} -\Delta u + (V_\infty + V(x))u = \left( \frac{1}{|x|^\alpha} * |u|^p \right) |u|^{p-2}u, \\ u \in H_0^1(\Omega), \end{cases}$$

where  $N \geq 3$ ,  $\alpha \in (0, N)$ ,  $p \in (\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2})$  and  $\Omega$  is an unbounded smooth domain in  $\mathbb{R}^N$  whose complement  $\mathbb{R}^N \setminus \Omega$  is bounded, possibly empty. We also assume that the potential  $V_\infty + V$  satisfies

$$(V_0) \quad V \in C^0(\mathbb{R}^N), \quad V_\infty \in (0, \infty), \quad \inf_{x \in \mathbb{R}^N} \{V_\infty + V(x)\} > 0, \quad \lim_{|x| \rightarrow \infty} V(x) = 0.$$

A special case of (1.1), relevant in physical applications, is the Choquard equation

$$(1.2) \quad -\Delta u + u = \left( \frac{1}{|x|} * |u|^2 \right) u, \quad u \in H^1(\mathbb{R}^3),$$

which models an electron trapped in its own hole, and was proposed by Choquard in 1976 as an approximation to Hartree-Fock theory of a one-component plasma [13]. This equation arises in many interesting situations related to the quantum theory of large systems of nonrelativistic bosonic atoms and molecules, see for example [10, 15] and the references therein. It was also proposed by Penrose in 1996 as a model for the self-gravitational collapse of a quantum mechanical wave-function [24]. In this context, problem (1.2) is usually called the nonlinear Schrödinger-Newton equation, see also [19, 20].

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In 1976 Lieb [13] proved the existence and uniqueness (modulo translations) of a minimizer to problem (1.2) by using symmetric decreasing rearrangement inequalities. Later, in [16], Lions showed the existence of infinitely many radially symmetric solutions to (1.2). Further results for related problems may be found in [1, 7, 8, 18, 22, 25, 26] and the references therein.

In 2010, Ma and Zhao [17] considered the generalized Choquard equation

$$(1.3) \quad -\Delta u + u = \left( \frac{1}{|x|^\alpha} * |u|^p \right) |u|^{p-2} u, \quad u \in H^1(\mathbb{R}^N),$$

and proved that, for  $p \geq 2$ , every positive solution of it is radially symmetric and monotone decreasing about some point, under the assumption that a certain set of real numbers, defined in terms of  $N$ ,  $\alpha$  and  $p$ , is nonempty. Under the same assumption, Cingolani, Clapp and Secchi [6] recently gave some existence and multiplicity results in the electromagnetic case, and established the regularity and some decay asymptotics at infinity of the ground states of (1.3). Moroz and van Schaftingen [21] eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters, and derived decay asymptotics at infinity for them, as well. These results will play an important role in our study.

In this article, we are interested in obtaining positive and sign changing solutions to problem (1.1). We study the case where both  $\Omega$  and  $V$  have some symmetries. If  $\Gamma$  is a closed subgroup of the group  $O(N)$  of linear isometries of  $\mathbb{R}^N$ , we denote by  $\Gamma x := \{gx : g \in \Gamma\}$  the  $\Gamma$ -orbit of  $x$ , by  $\#\Gamma x$  its cardinality, and by

$$\ell(\Gamma) := \min\{\#\Gamma x : x \in \mathbb{R}^N \setminus \{0\}\}.$$

We assume that  $\Omega$  and  $V$  are  $\Gamma$ -invariant, this means that  $\Gamma x \subset \Omega$  for every  $x \in \Omega$  and that  $V$  is constant on  $\Gamma x$  for each  $x \in \mathbb{R}^N$ . We consider a continuous group homomorphism  $\phi : \Gamma \rightarrow \mathbb{Z}/2$  and we look for solutions which satisfy

$$(1.4) \quad u(gx) = \phi(g)u(x) \quad \text{for all } g \in \Gamma \text{ and } x \in \Omega.$$

A function  $u$  with this property will be called  *$\phi$ -equivariant*. We denote by

$$G := \ker \phi.$$

Note that, if  $u$  satisfies (1.4), then  $u$  is  $G$ -invariant. Moreover,  $u(\gamma x) = -u(x)$  for every  $x \in \Omega$  and  $\gamma \in \phi^{-1}(-1)$ . Therefore, if  $\phi$  is an epimorphism (i.e. if it is surjective), every nontrivial solution to (1.1) which satisfies (1.4) changes sign. If  $\phi \equiv 1$  is the trivial homomorphism, then  $\Gamma = G$  and (1.4) simply says that  $u$  is  $G$ -invariant.

If  $Z$  is a  $\Gamma$ -invariant subset of  $\mathbb{R}^N$  and  $\phi$  is an epimorphism, the group  $\mathbb{Z}/2$  acts on the  $G$ -orbit space  $Z/G := \{Gx : x \in Z\}$  of  $Z$  as follows: we choose  $\gamma \in \Gamma$  such that  $\phi(\gamma) = -1$  and we define

$$(-1) \cdot Gx := G(\gamma x) \quad \text{for all } x \in Z.$$

This action is well defined and it does not depend on the choice of  $\gamma$ . We denote by

$$\Sigma := \{x \in \mathbb{R}^N : |x| = 1, \#\Gamma x = \ell(\Gamma)\}, \quad \Sigma_0 := \{x \in \Sigma : Gx = G(\gamma x)\}.$$

If  $Z$  is a nonempty  $\Gamma$ -invariant subset of  $\Sigma \setminus \Sigma_0$ , the action of  $\mathbb{Z}/2$  on its  $G$ -orbit space  $Z/G$  is free and the *Krasnoselskii genus* of  $Z/G$ , denoted  $\text{genus}(Z/G)$ , is defined to be the smallest  $k \in \mathbb{N}$  such that there exists a continuous map  $f : Z/G \rightarrow$

$\mathbb{S}^{k-1} := \{x \in \mathbb{R}^k : |x| = 1\}$  which is  $\mathbb{Z}/2$ -equivariant, i.e.  $f((-1) \cdot Gz) = -f(Gz)$  for every  $z \in Z$ . We define  $\text{genus}(\emptyset) := 0$ .

For each subgroup  $K$  of  $O(N)$  and each  $K$ -invariant subset  $Z$  of  $\mathbb{R}^N \setminus \{0\}$  we set

$$\mu(Kz) := \begin{cases} \inf\{|gz - hz| : g, h \in K, gz \neq hz\} & \text{if } \#Kz \geq 2, \\ 2|z| & \text{if } \#Kz = 1, \end{cases}$$

$$\mu_K(Z) := \inf_{z \in Z} \mu(Kz) \quad \text{and} \quad \mu^K(Z) := \sup_{z \in Z} \mu(Kz).$$

In the special case where  $K = G$  and  $Z = \Sigma$ , we simply write

$$\mu_G := \mu_G(\Sigma) \quad \text{and} \quad \mu^G := \mu^G(\Sigma).$$

We only consider the case  $\ell(\Gamma) < \infty$ , because if all  $\Gamma$ -orbits of  $\Omega$  are infinite it was already shown in [6, Theorem 1.1] that (1.1) has infinitely many solutions. In this case,  $\mu_G > 0$ .

We denote by  $c_\infty$  the energy of a ground state of the problem

$$\begin{cases} -\Delta u + V_\infty u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2} u, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

We shall look for solutions with small energy, i.e. which satisfy

$$(1.5) \quad \frac{p-1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha} dx dy < \ell(\Gamma) c_\infty.$$

In what follows, we assume that  $V$  satisfies  $(V_0)$  and we consider two cases: the case in which  $V$  is strictly negative at infinity, and that in which  $V$  takes on nonnegative values at infinity (which includes the case  $V = 0$ ). We shall prove the following results:

**Theorem 1.1.** *If  $p \geq 2$ ,  $\Omega$  is  $G$ -invariant and  $V$  is a  $G$ -invariant function which satisfies*

*(V<sub>1</sub>) There exist  $r_0 > 0$ ,  $c_0 > 0$  and  $\lambda \in (0, \mu^G \sqrt{V_\infty})$  such that*

$$V(x) \leq -c_0 e^{-\lambda|x|} \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq r_0,$$

*then (1.1) has at least one positive solution  $u$  which is  $G$ -invariant and satisfies (1.5) with  $\Gamma = G$ .*

**Theorem 1.2.** *If  $p \geq 2$ ,  $\Omega$  is  $\Gamma$ -invariant,  $\phi : \Gamma \rightarrow \mathbb{Z}/2$  is an epimorphism,  $Z$  is a  $\Gamma$ -invariant subset of  $\Sigma \setminus \Sigma_0$ ,  $V$  is a  $\Gamma$ -invariant function and the following holds:*

*(V<sub>2</sub>) There exist  $r_0 > 0$ ,  $c_0 > 0$  and  $\lambda \in (0, \mu_\Gamma(Z) \sqrt{V_\infty})$  such that*

$$V(x) \leq -c_0 e^{-\lambda|x|} \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq r_0,$$

*then problem (1.1) has at least  $\text{genus}(Z/G)$  pairs of sign changing solutions  $\pm u$ , which satisfy (1.4) and (1.5).*

**Theorem 1.3.** *If  $p \geq 2$ ,  $\ell(G) \geq 3$ ,  $\Omega$  is  $G$ -invariant and  $V$  is a  $G$ -invariant function which satisfies*

*(V<sub>3</sub>) There exist  $c_0 > 0$  and  $\kappa > \mu_G \sqrt{V_\infty}$  such that*

$$V(x) \leq c_0 e^{-\kappa|x|} \quad \text{for all } x \in \mathbb{R}^N,$$

*then (1.1) has at least one positive solution  $u$  which is  $G$ -invariant and satisfies (1.5) with  $\Gamma = G$ .*

**Theorem 1.4.** *If  $p \geq 2$ ,  $\Omega$  is  $\Gamma$ -invariant,  $\phi : \Gamma \rightarrow \mathbb{Z}/2$  is an epimorphism,  $Z$  is a  $\Gamma$ -invariant subset of  $\Sigma$ ,  $V$  is a  $\Gamma$ -invariant function and the following hold:*

*(Z<sub>0</sub>) There exists  $a_0 > 1$  such that*

$$\text{dist}(\gamma z, Gz) \geq a_0 \mu(Gz) \quad \text{for all } z \in Z \text{ and } \gamma \in \Gamma \setminus G.$$

*(V<sub>4</sub>) There exist  $c_0 > 0$  and  $\kappa > \mu^\Gamma(Z)\sqrt{V_\infty}$  such that*

$$V(x) \leq c_0 e^{-\kappa|x|} \quad \text{for all } x \in \mathbb{R}^N,$$

*then (1.1) has at least  $\text{genus}(Z/G)$  pairs of sign changing solutions  $\pm u$ , which satisfy (1.4) and (1.5).*

Theorem 1.1 was proved in [6] for  $\Omega = \mathbb{R}^N$ , under additional assumptions on  $\alpha$  and  $p$ . As far as we know, Theorem 1.3 is the first existence result for potentials  $V$  which are nontrivial and take nonnegative values at infinity. In the local case, Bahri and Lions proved existence for this type of potentials without any symmetries [2]. Unfortunately, some of the facts used in their proof are not available in the nonlocal case.

As we mentioned before, the existence of infinitely many solutions is known in the radial case [16] and in the case where every  $G$ -orbit in  $\Omega$  is infinite [6]. In contrast, Theorems 1.2 and 1.4 provide multiple solutions when the data have only finite symmetries. The following examples, which illustrate these results, are taken from [9], where similar results for the local case were recently obtained.

**Example 1.** Let  $\Gamma$  be the group spanned by the reflection  $\gamma : \mathbb{R}^N \rightarrow \mathbb{R}^N$  on a linear subspace  $W$  of  $\mathbb{R}^N$ . If  $\Omega$  and  $V$  are invariant under this reflection, we may take  $\phi : \Gamma \rightarrow \mathbb{Z}/2$  to be the epimorphism given by  $\phi(\gamma) := -1$  and  $Z$  to be the unit sphere in the orthogonal complement of  $W$ . Then, Theorem 1.2 yields

$$\text{genus}(Z) = N - \dim W$$

pairs of solutions to problem (1.1) provided (V<sub>2</sub>) holds for some  $\lambda \in (0, 2\sqrt{V_\infty})$ .

**Example 2.** If  $N = 2n$  we identify  $\mathbb{R}^N$  with  $\mathbb{C}^n$  and take  $\Gamma$  to be the cyclic group of order  $2m$  spanned by  $\rho(z_1, \dots, z_n) := (e^{\pi i/m} z_1, \dots, e^{\pi i/m} z_n)$  and  $\phi : \Gamma \rightarrow \mathbb{Z}/2$  to be the epimorphism given by  $\phi(\rho) := -1$ . Then  $G := \ker \phi$  is the cyclic subgroup of order  $m$  spanned by  $\rho^2$ ,  $\Sigma = \mathbb{S}^{N-1}$  and  $\Sigma_0 = \emptyset$ . So we may take  $Z := \mathbb{S}^{N-1}$ . The genus of  $\mathbb{S}^{N-1}/G$  can be estimated in many cases. For example, if  $m = 2^k$ , Lemma 6.1 in [9] together with Theorem 1.2 in [3] give

$$\text{genus}(\mathbb{S}^{N-1}/G) \geq \frac{N-1}{2^k} + 1.$$

Since  $\mu_\Gamma(\mathbb{S}^{N-1}) = |e^{\pi i/m} - 1|$ , if condition (V<sub>2</sub>) holds for  $m = 2^k$ , it will also hold for  $m = 2^j$  with  $0 \leq j < k$ . An easy argument shows that, if  $u_j$  satisfies (1.4) for  $m = 2^j$ ,  $u_l$  satisfies (1.4) for  $m = 2^l$  and  $j \neq l$ , then  $u_j \neq u_l$ , see [9, section 1]. Therefore, Theorem 1.2 provides at least

$$\sum_{j=0}^k \frac{N-1}{2^j} + k + 1 = (N-1) \frac{2^{k+1}-1}{2^k} + k + 1$$

pairs of sign changing solutions in this case.

The group  $G$  in the previous example satisfies  $\ell(G) = m$ . This shows that there are many groups satisfying the symmetry assumption in Theorem 1.3 when  $N$  is even. If  $N$  is odd not many groups satisfy  $\ell(G) \geq 3$ . For example, if  $N = 3$ , the only subgroups of  $O(3)$  which satisfy this condition are the rotation groups of the icosahedron, octahedron and tetrahedron,  $I$ ,  $O$  and  $T$ , and the groups  $I \times \mathbb{Z}_2^5$ ,  $O \times \mathbb{Z}_2^5$ ,  $T \times \mathbb{Z}_2^5$  and  $O^-$  described in [5, Appendix A].

Note that  $(Z_0)$  implies that  $Z \subset \Sigma \setminus \Sigma_0$ . Condition  $(Z_0)$  cannot be realized if  $N = 3$ . Next, we give an example for which  $(Z_0)$  holds.

**Example 3.** We identify  $\mathbb{R}^{4n}$  with  $\mathbb{C}^n \times \mathbb{C}^n$  and consider the subgroup  $\Gamma$  of  $O(4n)$  spanned by  $\rho$  and  $\gamma$ , where  $\rho(y, z) := (e^{\pi i/m} y, e^{\pi i/m} z)$  and  $\gamma(y, z) := (-\bar{z}, \bar{y})$  for  $(y, z) \in \mathbb{C}^n \times \mathbb{C}^n$  and some  $m \geq 3$ . We define  $\phi : \Gamma \rightarrow \mathbb{Z}/2$  by  $\phi(\rho) = 1$ ,  $\phi(\gamma) = -1$ . Then  $G := \ker \phi$  is the cyclic subgroup of order  $2m$  spanned by  $\rho$ . Since  $m \geq 3$ , property  $(Z_0)$  holds for  $Z := \mathbb{S}^{4n-1}$ . We showed in [9, Proposition 6.1] that  $\text{genus}(\mathbb{S}^{4n-1}/G) \geq 2n + 1$ . Consequently, if  $\Omega$  and  $V$  are  $\Gamma$ -invariant and  $(V_4)$  holds, Theorem 1.4 yields  $2n + 1$  pairs of sign changing solutions to problem (1.1). Note that  $\mu^G(\mathbb{S}^{4n-1}) = |e^{\pi i/m} - 1|$ , hence  $(V_4)$  becomes less restrictive as  $m$  increases.

This paper is organized as follows: In section 2 we set the variational framework for problem (1.1). In section 3 some preliminary asymptotic estimates are established. In section 4 we consider potentials which are strictly negative at infinity and prove Theorems 1.1 and 1.2. Finally, in section 5 we consider potentials which take on nonnegative values at infinity and prove Theorems 1.3 and 1.4.

## 2. THE VARIATIONAL SETTING

From now on we shall assume without loss of generality that  $V_\infty = 1$ . Assumption  $(V_0)$  guarantees that

$$(2.1) \quad \langle u, v \rangle_V := \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} (1 + V(x)) uv$$

is a scalar product in  $H_0^1(\Omega)$  and that the induced norm

$$(2.2) \quad \|u\|_V := \left( \int_{\Omega} (|\nabla u|^2 + (1 + V(x)) u^2) \right)^{1/2}$$

is equivalent to the usual one. If  $V = 0$  we write  $\langle u, v \rangle$  and  $\|u\|$  instead of  $\langle u, v \rangle_0$  and  $\|u\|_0$ .

As usual, we identify  $u \in H_0^1(\Omega)$  with its extension to  $\mathbb{R}^N$  obtained by setting  $u = 0$  in  $\mathbb{R}^N \setminus \Omega$ . We define

$$\mathbb{D}(u) := \int_{\Omega} \left( \frac{1}{|x|^\alpha} * |u|^p \right) |u|^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x - y|^\alpha} dx dy$$

and set  $r := \frac{2N}{2N-\alpha}$ . As  $p \in (\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2})$ , one has that  $pr \in (2, \frac{2N}{N-2})$ . The Hardy-Littlewood-Sobolev inequality [14, Theorem 4.3] implies the existence of a positive constant  $\bar{C}$  such that

$$(2.3) \quad \mathbb{D}(u) \leq \bar{C} |u|_{pr}^{2p} \quad \text{for all } u \in H^1(\mathbb{R}^N),$$

where  $|u|_q := (\int_{\mathbb{R}^N} |u|^q)^{1/q}$  is the norm in  $L^q(\mathbb{R}^N)$ . This shows that  $\mathbb{D}$  is well defined in  $H^1(\mathbb{R}^N)$ .

We shall assume from now on that  $p \in [2, \frac{2N-\alpha}{N-2})$ . Then the functional  $J_V : H_0^1(\Omega) \rightarrow \mathbb{R}$  given by

$$(2.4) \quad J_V(u) := \frac{1}{2} \|u\|_V^2 - \frac{1}{2p} \mathbb{D}(u)$$

is of class  $\mathcal{C}^2$ . Its derivative is

$$J'_V(u)v := \langle u, v \rangle_V - \int_{\Omega} \left( \frac{1}{|x|^\alpha} * |u|^p \right) |u|^{p-2} uv.$$

Hence, the solutions to problem (1.1) are the critical points of  $J_V$ .

The homomorphism  $\phi : \Gamma \rightarrow \mathbb{Z}/2$  induces an orthogonal action of  $\Gamma$  on  $H_0^1(\Omega)$  as follows: for  $\gamma \in \Gamma$  and  $u \in H_0^1(\Omega)$  we define  $\gamma u \in H_0^1(\Omega)$  by

$$(\gamma u)(x) := \phi(\gamma) u(\gamma^{-1}x).$$

Since  $\langle \gamma u, \gamma v \rangle_V = \langle u, v \rangle_V$  and  $\mathbb{D}(\gamma u) = \mathbb{D}(u)$  for all  $\gamma \in \Gamma$ ,  $u, v \in H_0^1(\Omega)$ , the functional  $J_V$  is  $\Gamma$ -invariant. By the principle of symmetric criticality [23, 27] the critical points of the restriction of  $J_V$  to the fixed point space of this action, which we denote by

$$\begin{aligned} H_0^1(\Omega)^\phi &:= \{u \in H_0^1(\Omega) : \gamma u = u \ \forall \gamma \in \Gamma\} \\ &= \{u \in H_0^1(\Omega) : u(\gamma x) = \phi(\gamma) u(x) \ \forall \gamma \in \Gamma, \forall x \in \Omega\}, \end{aligned}$$

are the solutions to problem (1.1) that satisfy (1.4). The nontrivial ones lie on the Nehari manifold

$$\mathcal{N}_{\Omega, V}^\phi := \{u \in H_0^1(\Omega)^\phi : u \neq 0, \|u\|_V^2 = \mathbb{D}(u)\},$$

which is of class  $\mathcal{C}^2$  and radially diffeomorphic to the unit sphere in  $H_0^1(\Omega)^\phi$ . The radial projection  $\pi : H_0^1(\Omega)^\phi \setminus \{0\} \rightarrow \mathcal{N}_{\Omega, V}^\phi$  is given by

$$(2.5) \quad \pi(u) := \left( \frac{\|u\|_V^2}{\mathbb{D}(u)} \right)^{\frac{1}{2(p-1)}} u.$$

Accordingly, for every  $u \in H_0^1(\Omega)^\phi \setminus \{0\}$ ,

$$(2.6) \quad J_V(\pi(u)) = \frac{p-1}{2p} \left( \frac{\|u\|_V^2}{\mathbb{D}(u)^{\frac{1}{p}}}} \right)^{\frac{p}{p-1}}.$$

We set

$$c_{\Omega, V}^\phi := \inf_{\mathcal{N}_{\Omega, V}^\phi} J_V.$$

If  $\phi \equiv 1$  is the trivial homomorphism, then  $\Gamma = G := \ker \phi$ . In this case we shall write  $H_0^1(\Omega)^G$ ,  $\mathcal{N}_{\Omega, V}^G$  and  $c_{\Omega, V}^G$  instead of  $H_0^1(\Omega)^\phi$ ,  $\mathcal{N}_{\Omega, V}^\phi$  and  $c_{\Omega, V}^\phi$ . If  $G = \{1\}$  is the trivial group, we shall omit it from the notation and write simply  $H_0^1(\Omega)$ ,  $\mathcal{N}_{\Omega, V}$  and  $c_{\Omega, V}$ .

The problem

$$(2.7) \quad \begin{cases} -\Delta u + u = \left( \frac{1}{|x|^\alpha} * |u|^p \right) |u|^{p-2} u, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

plays a special role: it is the limit problem for (1.1). In this case we write  $J_\infty$ ,  $\mathcal{N}_\infty$  and  $c_\infty$  instead of  $J_0$ ,  $\mathcal{N}_{\mathbb{R}^N, 0}$  and  $c_{\mathbb{R}^N, 0}$ .

It is known that  $c_\infty$  is attained at a positive function  $\omega \in H^1(\mathbb{R}^N)$  (see for example [21, Theorem 3]). The following result shows, however, that  $c_{\Omega,V}^\phi$  is not necessarily attained. We write

$$B_r(\xi) := \{x \in \mathbb{R}^N : |x - \xi| < r\}.$$

**Proposition 2.1.** *If  $V \geq 0$ , then  $c_{\Omega,V} = c_\infty$ . If, additionally,  $V \not\equiv 0$  when  $\Omega = \mathbb{R}^N$ , then  $c_{\Omega,V}$  is not attained.*

*Proof.* Since  $H_0^1(\Omega) \subset H^1(\mathbb{R}^N)$  and  $V \geq 0$  one easily concludes that  $c_{\Omega,V} \geq c_\infty$ . Let  $R > 0$  be such that  $(\mathbb{R}^N \setminus \Omega) \subset B_R(0)$ , and let  $(x_n)$  be a sequence in  $\mathbb{R}^N$  such that  $|x_n| > R$  and  $|x_n| \rightarrow \infty$ . We choose a cut-off function  $\chi \in C_c^\infty(\mathbb{R}^N)$  such that  $0 \leq \chi \leq 1$ ,  $\chi(x) = 1$  if  $|x| \leq 1$  and  $\chi(x) = 0$  if  $|x| \geq 2$ . We define  $r_n := \frac{1}{2}(|x_n| - R)$  and

$$u_n(x) := \chi\left(\frac{x - x_n}{r_n}\right)\omega(x - x_n).$$

Then  $u_n \in H_0^1(\Omega)$ ,  $u_n \neq 0$ ,  $u_n \rightharpoonup 0$  weakly in  $H_0^1(\mathbb{R}^N)$  and  $u_n \rightarrow 0$  strongly in  $L_{loc}^2(\mathbb{R}^N)$ . An easy argument shows that

$$\lim_{n \rightarrow \infty} \|u_n\|_V^2 = \lim_{n \rightarrow \infty} \|u_n\|^2 = \|\omega\|^2.$$

Applying the Lebesgue dominated convergence theorem, we obtain that

$$\lim_{n \rightarrow \infty} \mathbb{D}(u_n) = \mathbb{D}(\omega).$$

Consequently, from (2.6) we obtain that  $J_V(\pi(u_n)) \rightarrow J_\infty(\omega) = c_\infty$ . Therefore  $c_{\Omega,V} \leq c_\infty$ , and hence  $c_{\Omega,V} = c_\infty$ .

Now, if there were  $u \in \mathcal{N}_{\Omega,V}$  satisfying  $J_V(u) = c_{\Omega,V}$ , then  $u$  would be a nontrivial solution of problem (2.7) with minimum energy and  $\|u\|_V^2 = \|u\|^2$ . We distinguish two cases: (1) If  $\Omega = \mathbb{R}^N$  then, by assumption,  $V$  is strictly positive on some open set  $U$  of  $\mathbb{R}^N$ . Since

$$0 = \|u\|_V^2 - \|u\|^2 = \int_{\mathbb{R}^N} V(x)u^2 \geq \int_U V(x)u^2 \geq 0,$$

we conclude that  $u = 0$  in  $U$ . (2) If  $\Omega \neq \mathbb{R}^N$  then  $u = 0$  in  $\mathbb{R}^N \setminus \Omega$ . In both cases, we obtain a contradiction to the unique continuation principle [11, 12]. As a result,  $c_{\Omega,V}$  is not attained.  $\square$

We say that  $J_V$  satisfies condition  $(PS)_c^\phi$  if every sequence  $(u_n)$  such that

$$(2.8) \quad u_n \in H_0^1(\Omega)^\phi, \quad J_V(u_n) \rightarrow c, \quad J'_V(u_n) \rightarrow 0 \text{ in } H^{-1}(\Omega),$$

has a convergent subsequence in  $H_0^1(\Omega)$ . If  $\phi \equiv 1$ , we write  $(PS)_c^G$  instead of  $(PS)_c^\phi$ .

**Proposition 2.2.**  *$J_V$  satisfies condition  $(PS)_c^\phi$  for all*

$$c < \ell(\Gamma)c_\infty.$$

*Proof.* This follows from Proposition 3.1 in [6] taking  $A = 0$ ,  $G = \Gamma$ ,  $\tau = \phi$  (notice that  $\mathbb{Z}/2$  is a subgroup of  $\mathbb{S}^1$ ) and  $u_n \in H_0^1(\Omega)^\phi \subset H^1(\mathbb{R}^N, \mathbb{C})^\phi$ .  $\square$

We denote by  $\nabla J_V$  the gradient of  $J_V$  with respect to the scalar product (2.1), and by  $\nabla_{\mathcal{N}} J_V(u)$  the orthogonal projection of  $\nabla J_V(u)$  onto the tangent space  $T_u \mathcal{N}_{\Omega,V}^\phi$  to the Nehari manifold  $\mathcal{N}_{\Omega,V}^\phi$  at the point  $u \in \mathcal{N}_{\Omega,V}^\phi$ . We shall say that  $J_V$  satisfies condition  $(PS)_c^\phi$  on  $\mathcal{N}_{\Omega,V}^\phi$  if every sequence  $(u_n)$  such that

$$(2.9) \quad u_n \in \mathcal{N}_{\Omega,V}^\phi, \quad J_V(u_n) \rightarrow c, \quad \nabla_{\mathcal{N}} J_V(u_n) \rightarrow 0,$$

contains a convergent subsequence in  $H_0^1(\Omega)$ .

**Corollary 2.3.**  $J_V$  satisfies condition  $(PS)_c^\phi$  on  $\mathcal{N}_{\Omega,V}^\phi$  for all

$$c < \ell(\Gamma)c_\infty.$$

*Proof.* The proof is completely analogous to that of Corollary 3.8 in [9].  $\square$

### 3. ASYMPTOTIC ESTIMATES

The ground states of problem (2.7) have been recently studied in [6, 21]. The following result holds true.

**Theorem 3.1.** *Let  $\omega$  be a ground state of problem (2.7). Then  $\omega \in L^1(\mathbb{R}^N) \cap \mathcal{C}^\infty(\mathbb{R}^N)$ ,  $\omega$  does not change sign and it is radially symmetric and monotone decreasing in the radial direction with respect to some fixed point. Moreover,  $\omega$  has the following asymptotic behavior:*

(i) *If  $p > 2$  then*

$$\lim_{|x| \rightarrow \infty} |\omega(x)| |x|^{\frac{N-1}{2}} e^{|x|} \in (0, \infty).$$

(ii) *If  $p = 2$  then*

$$\lim_{|x| \rightarrow \infty} |\omega(x)| |x|^{\frac{N-1}{2}} e^{Q(|x|)} \in (0, \infty),$$

where

$$Q(t) := \int_\delta^t \sqrt{1 - \frac{\delta^\alpha}{s^\alpha}} ds \quad \text{and} \quad \delta^\alpha := (4 - \alpha)c_\infty.$$

*Proof.* See Theorems 3 and 4 in [21]. Note that  $\omega$  is a solution of (2.7) if and only if  $u := \lambda^{-\frac{1}{2(p-1)}} \omega$  is a solution of problem (1.1) in [21], where  $\lambda := \frac{\Gamma(\alpha/2)}{\Gamma((N-\alpha)/2)\pi^{N/2}2^{N-\alpha}}$  and  $\Gamma$  denotes here (and only here) the gamma function (and not the group).  $\square$

In what follows,  $\omega$  will denote a positive ground state of problem (2.7) which is radially symmetric with respect to the origin. We continue to assume that  $p \geq 2$ .

**Lemma 3.2.**

$$\lim_{|x| \rightarrow \infty} \omega(x) |x|^{\frac{N-1}{2}} e^{a|x|} = \begin{cases} \infty & \text{if } a > 1, \\ 0 & \text{if } a \in (0, 1). \end{cases}$$

*Proof.* Set  $b := \frac{N-1}{2}$ . We shall prove this result for  $p = 2$ . The proof for  $p > 2$  is an immediate consequence of Theorem 3.1. Observe that, for every  $\nu \in (0, 1)$  it holds true that

$$\sqrt{1 - \frac{\delta^\alpha}{s^\alpha}} \leq 1 \quad \text{if } s \geq \delta \quad \text{and} \quad \sqrt{1 - \frac{\delta^\alpha}{s^\alpha}} \geq \nu \quad \text{if } s \geq \frac{\delta}{(1 - \nu^2)^{1/\alpha}} =: s_\nu,$$

and, hence, that

$$Q(t) \leq t \quad \text{if } t \geq \delta \quad \text{and} \quad \nu(t - s_\nu) \leq Q(t) \quad \text{if } t \geq s_\nu.$$

Consequently, if  $|x| \geq \delta$  then

$$\omega(x) |x|^b e^{a|x|} = \omega(x) |x|^b e^{Q(|x|)} e^{a|x| - Q(|x|)} \geq \omega(x) |x|^b e^{Q(|x|)} e^{(a-1)|x|}.$$

If  $a > 1$ , the conclusion follows from Theorem 3.1. If  $a \in (0, 1)$ , we fix  $\nu \in (a, 1)$ . Then, for all  $|x| \geq s_\nu$ ,

$$\omega(x)|x|^b e^{a|x|} = \omega(x)|x|^b e^{Q(|x|)} e^{a|x|-Q(|x|)} \leq \omega(x)|x|^b e^{Q(|x|)} e^{(a-\nu)|x|+\nu s_\nu},$$

and using once more Theorem 3.1 the conclusion follows.  $\square$

For  $\zeta \in \mathbb{R}^N$  we set

$$(3.1) \quad \omega_\zeta(x) := \omega(x - \zeta).$$

**Lemma 3.3.** *For each  $a \in (0, 1)$ ,*

$$\lim_{|\zeta| \rightarrow \infty} \int_{\mathbb{R}^N} \omega^{p-1} \omega_\zeta |\zeta|^{\frac{N-1}{2}} e^{a|\zeta|} = 0.$$

*Proof.* By Lemma 3.2 we have that, for each  $\nu \in (0, 1)$ , there exists a constant  $C_\nu > 0$  such that

$$\omega(x) \leq C_\nu e^{-\nu|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

We fix  $\nu_1, \nu_2 \in (a, 1)$  with  $\nu_1 < \nu_2$ . In what follows,  $C$  will denote different positive constants depending only on  $\nu_1$  and  $\nu_2$ . We have that

$$\begin{aligned} \int_{\mathbb{R}^N} \omega^{p-1} \omega_\zeta &\leq C \int_{\mathbb{R}^N} e^{-\nu_1(p-1)|x|} e^{-\nu_2|x-\zeta|} dx \leq C \int_{\mathbb{R}^N} e^{-\nu_1|x|} e^{-\nu_2|x-\zeta|} dx \\ &= C \int_{\mathbb{R}^N} e^{-\nu_1(|x|+|x-\zeta|)} e^{-(\nu_2-\nu_1)|x-\zeta|} dx \leq C e^{-\nu_1|\zeta|} \int_{\mathbb{R}^N} e^{-(\nu_2-\nu_1)|x|} dx \\ &= C e^{-\nu_1|\zeta|}. \end{aligned}$$

Therefore,

$$0 \leq \int_{\mathbb{R}^N} \omega^{p-1} \omega_\zeta |\zeta|^{\frac{N-1}{2}} e^{a|\zeta|} \leq C |\zeta|^{\frac{N-1}{2}} e^{-(\nu_1-a)|\zeta|},$$

which implies the result.  $\square$

For  $\zeta \in \mathbb{R}^N$  we define

$$(3.2) \quad I(\zeta) := \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} * \omega^p \right) \omega^{p-1} \omega_\zeta.$$

**Lemma 3.4.** *For each  $a \in (0, 1)$ ,*

$$\lim_{|\zeta| \rightarrow \infty} I(\zeta) |\zeta|^{\frac{N-1}{2}} e^{a|\zeta|} = 0.$$

*Proof.* As  $p < \frac{2N-\alpha}{N-2}$ , we have that  $\frac{N-\alpha}{N} \left(1 - \frac{1}{p}\right) - \frac{2}{N} < \frac{N-\alpha}{Np}$ . By [21, Section 4, Claim 1],  $|u|^p \in L^{\frac{N}{N-\alpha}}(\mathbb{R}^N)$ . Hence,  $\frac{1}{|x|^\alpha} * \omega^p \in L^\infty(\mathbb{R}^N)$ , cf. [14, Section 4.3 (9)]. Thus,

$$0 \leq I(\zeta) |\zeta|^{\frac{N-1}{2}} e^{a|\zeta|} \leq C \int_{\mathbb{R}^N} \omega^{p-1} \omega_\zeta |\zeta|^{\frac{N-1}{2}} e^{a|\zeta|}.$$

From Lemma 3.3 we obtain the conclusion.  $\square$

**Lemma 3.5.** *For every  $a > 1$ , there exists a positive constant  $k_a$  such that*

$$I(\zeta) |\zeta|^{\frac{N-1}{2}} e^{a|\zeta|} \geq k_a \quad \text{for all } |\zeta| \geq 1.$$

*Proof.* Set  $b := \frac{N-1}{2}$ . Lemma 3.2 asserts the existence of positive constants  $C_a, R_a$  such that  $C_a|x|^{-b}e^{-a|x|} \leq \omega(x)$  if  $|x| \geq R_a$ . Let  $C_1 > 0$  be such that  $\omega(x) \geq C_1e^{-a|x|}$  for all  $|x| \leq R_a$ . Setting  $C_2 := \min\{C_a, C_1\}$  we conclude that

$$\omega(x) \geq C_2(1 + |x|)^{-b}e^{-a|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

Hence,

$$\begin{aligned} \omega(x - \zeta)|\zeta|^b e^{a|\zeta|} &\geq C_2(1 + |x - \zeta|)^{-b}e^{-a|x - \zeta|}|\zeta|^b e^{a|\zeta|} \\ &\geq C_2(1 + |x - \zeta|)^{-b}|\zeta|^b e^{-a|x|} \quad \text{for } x, \zeta \in \mathbb{R}^N. \end{aligned}$$

Note that, if  $|x| \leq 1 \leq |\zeta|$ , then  $1 + |x - \zeta| \leq 1 + |x| + |\zeta| \leq 3|\zeta|$  and so

$$\omega(x - \zeta)|\zeta|^b e^{a|\zeta|} \geq C_3 e^{-a|x|} \quad \text{for } x, \zeta \in \mathbb{R}^N \text{ with } |x| \leq 1 \leq |\zeta|,$$

where  $C_3 := 3^{-b}C_2$ . Consequently,

$$\begin{aligned} I(\zeta)|\zeta|^b e^{a|\zeta|} &= \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} * \omega^p \right) (x) \omega^{p-1}(x) \omega(x - \zeta) |\zeta|^b e^{a|\zeta|} dx \\ &\geq C_3 \int_{|x| \leq 1} \left( \frac{1}{|x|^\alpha} * \omega^p \right) (x) \omega^{p-1}(x) e^{-a|x|} =: k_a \quad \text{for } |\zeta| \geq 1, \end{aligned}$$

as claimed.  $\square$

**Remark 3.6.** As in the local case (see [9, section 5]) it is possible to prove that, for  $p > 2$ , there exists a positive constant  $k_1$  such that

$$\lim_{|\xi| \rightarrow \infty} I(\xi) |\xi|^{\frac{N-1}{2}} e^{|\xi|} = k_1.$$

However, we will not need this fact.

For  $\zeta \in \mathbb{R}^N$  we define

$$(3.3) \quad A(\zeta) := \int_{\mathbb{R}^N} V^+(x) \omega^2(x - \zeta) dx.$$

**Lemma 3.7.** Let  $M \in (0, 2)$ . If  $V(x) \leq ce^{-\iota|x|}$  for all  $x \in \mathbb{R}^N$  with  $c > 0$  and  $\iota > M$ , then

$$\lim_{|\zeta| \rightarrow \infty} A(\zeta) |\zeta|^{\frac{N-1}{2}} e^{M|\zeta|} = 0.$$

*Proof.* See [9, Lemma 5.2].  $\square$

**Lemma 3.8.** If  $f \in C_c^0(\mathbb{R}^N)$ ,  $q > 1$  and  $a \in (0, 1)$ , then

$$\lim_{|\zeta| \rightarrow \infty} \left( \int_{\mathbb{R}^N} f(x) \omega^q(x - \zeta) dx \right) |\zeta|^{\frac{N-1}{2}} e^{qa|\zeta|} = 0.$$

*Proof.* Set  $b := \frac{N-1}{2}$ . Let  $T > 0$  be such that  $\text{supp}(f) \subset B_T(0)$ . By Lemma 3.2 there exists  $C > 0$  such that

$$\omega(x) \leq C(T + |x|)^{-b}e^{-a|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

Therefore, if  $|x| \leq T$ ,

$$\begin{aligned} \omega^q(x - \zeta) |\zeta|^b e^{qa|\zeta|} &\leq C^q (T + |x - \zeta|)^{-qb} e^{-qa|x - \zeta|} |\zeta|^b e^{qa|\zeta|} \\ &\leq C^q (|x| + |x - \zeta|)^{-qb} e^{-qa|x - \zeta|} |\zeta|^b e^{qa|\zeta|} \leq C^q |\zeta|^{(1-q)b} e^{qa|x|}. \end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^N} |f(x)| \omega^q(x - \zeta) |\zeta|^b e^{qa|\zeta|} dx \leq C^q |\zeta|^{(1-q)b} \int_{|x| \leq T} |f(x)| e^{qa|x|} dx =: C_1 |\zeta|^{(1-q)b},$$

from which the assertion of Lemma 3.8 follows.  $\square$

#### 4. PROOF OF THEOREMS 1.1 AND 1.2

Let  $Z$  be a  $\Gamma$ -invariant subset of  $\Sigma$  and let  $\lambda \in (0, \mu_\Gamma(Z))$  be such that  $(V_2)$  holds (recall that we are assuming that  $V_\infty = 1$ ). We choose  $\nu \in (0, 1)$  such that  $\lambda \in (0, \mu_\Gamma(Z)\nu)$ ,  $\varepsilon \in (0, \frac{\mu_\Gamma(Z)\nu - \lambda}{\mu_\Gamma(Z)\nu + \lambda})$  and a radially symmetric cut-off function  $\chi \in C^\infty(\mathbb{R}^N)$  such that  $0 \leq \chi \leq 1$ ,  $\chi(x) = 1$  if  $|x| \leq 1 - \varepsilon$  and  $\chi(x) = 0$  if  $|x| \geq 1$ . Let  $\omega \in H^1(\mathbb{R}^N)$  be a positive ground state of problem (2.7) which is radially symmetric about the origin. For  $S > 0$  we define  $\omega^S \in H^1(\mathbb{R}^N)$  by

$$\omega^S(x) := \chi\left(\frac{x}{S}\right) \omega(x).$$

Lemma 3.2 allows to obtain the following asymptotic estimates:

$$\left| \|\omega\|^2 - \|\omega^S\|^2 \right| = O(e^{-2\nu(1-\varepsilon)S}), \quad |\mathbb{D}(\omega) - \mathbb{D}(\omega^S)| = O(e^{-p\nu(1-\varepsilon)S})$$

as  $S \rightarrow \infty$ , see [6, Lemma 4.1]. We set  $\rho := \frac{\mu_\Gamma(Z)\nu + \lambda}{4\nu}$ , and for every  $z \in Z$  we consider the function

$$v_{R,z}(x) := \omega^{\rho R}(x - Rz).$$

Note that  $\text{supp}(v_{R,z}) \subset \overline{B_{\rho R}(Rz)}$ . Note also that  $\rho \in (0, 1)$  because  $\mu_\Gamma(Z) \leq 2$ . Therefore, since  $\mathbb{R}^N \setminus \Omega$  is bounded, there exists  $R_0 > 0$  such that  $v_{R,z} \in H_0^1(\Omega)$  for all  $z \in Z$  and  $R \geq R_0$ .

**Lemma 4.1.** *There exist  $d_0 > 0$  and  $\varrho_0 > R_0$  such that  $v_{R,z} \in H_0^1(\Omega)$  and*

$$J_V(\pi(v_{R,z})) \leq c_\infty - d_0 e^{-\lambda R} \quad \text{for all } z \in Z \text{ and } R \geq \varrho_0.$$

*Proof.* This is a special case of [6, Lemma 4.2] with  $A = 0$ .  $\square$

Let  $\phi : \Gamma \rightarrow \mathbb{Z}/2$  be a continuous group homomorphism and set  $G := \ker \phi$ . We fix  $R \geq \varrho_0$ , and for  $z \in Z$  we define

$$(4.1) \quad \theta(z) := \sum_{gz \in \Gamma z} \phi(g) v_{R,gz}.$$

**Proposition 4.2.** *If either  $\phi \equiv 1$  or  $Z \subset \Sigma \setminus \Sigma_0$ , then  $\theta(z)$  is well defined.  $\theta(z)$  is  $\phi$ -equivariant and*

$$J_V(\pi(\theta(z))) \leq \ell(\Gamma) (c_\infty - d_0 e^{-\lambda R}) \quad \text{for all } z \in Z.$$

*If moreover  $Z \neq \emptyset$ , then  $c_{\Omega,V}^\phi < \ell(\Gamma)c_\infty$ .*

*Proof.* Let  $z \in Z$ . If  $g_1, g_2 \in \Gamma$  are such that  $g_1 z = g_2 z$ , then  $g_2^{-1} g_1 z = z$ . Hence, if either  $\phi \equiv 1$  or  $z \notin \Sigma_0$ , it must be true that  $\phi(g_2^{-1} g_1) = 1$ . Thus  $\phi(g_1) = \phi(g_2)$ . This shows that  $\theta(z)$  is well defined. It is clearly  $\phi$ -equivariant.

On the other hand, since  $|Rg_1 z - Rg_2 z| \geq R\mu_\Gamma(Z) > 2\rho R$  when  $g_1 z \neq g_2 z$ , we have that  $\text{supp}(v_{R,g_1 z}) \cap \text{supp}(v_{R,g_2 z}) = \emptyset$ . Consequently,  $\|\theta(z)\|_V^2 = \ell(\Gamma) \|v_{R,z}\|_V^2$

and  $\mathbb{D}(\theta(z)) > \ell(\Gamma)\mathbb{D}(v_{R,z})$ . From (2.6) and Lemma 4.1 we obtain

$$\begin{aligned} J_V(\pi(\theta(z))) &\leq \frac{p-1}{2p} \left( \frac{\ell(\Gamma)\|v_{R,z}\|_V^2}{[\ell(\Gamma)\mathbb{D}(v_{R,z})]^{\frac{1}{p}}} \right)^{\frac{p}{p-1}} \\ &= \ell(\Gamma)J_V(\pi(v_{R,z})) \leq \ell(\Gamma)(c_\infty - d_0e^{-\lambda R}). \end{aligned}$$

Finally, since  $\pi(\theta(z)) \in \mathcal{N}_{\Omega,V}^\phi$ , we conclude that  $c_{\Omega,V}^\phi < \ell(\Gamma)c_\infty$ .  $\square$

*Proof of Theorem 1.1.* Let  $\phi \equiv 1$ , so that  $\Gamma = G$ . If assumption  $(V_1)$  holds for  $\lambda \in (0, \mu^G)$ , we choose  $\zeta \in \Sigma$  such that  $\mu(G\zeta) \in (\lambda, \mu^G]$  and define  $Z := G\zeta$ . Thus  $\mu_G(Z) = \mu(G\zeta)$  and assumption  $(V_2)$  holds for  $\lambda \in (0, \mu_G(Z))$ . Hence, we may apply Proposition 4.2 to these data to conclude that  $c_{\Omega,V}^G < \ell(G)c_\infty$ . Corollary 2.3 then asserts that  $J_V$  satisfies condition  $(PS)_c^G$  on  $\mathcal{N}_{\Omega,V}^G$  for  $c := c_{\Omega,V}^G$ . Therefore, there exists  $u \in \mathcal{N}_{\Omega,V}^G$  such that  $J_V(u) = c_{\Omega,V}^G$ . Finally, observe that  $|u| \in \mathcal{N}_{\Omega,V}^G$  and  $J_V(|u|) = J_V(u)$ . Hence problem (1.1) has a  $G$ -invariant positive solution  $|u|$  satisfying  $J_V(|u|) < \ell(G)c_\infty$ .  $\square$

*Proof of Theorem 1.2.*  $\mathcal{N}_{\Omega,V}^\phi$  is a  $\mathcal{C}^2$ -manifold and  $J_V : \mathcal{N}_{\Omega,V}^\phi \rightarrow \mathbb{R}$  is an even  $\mathcal{C}^2$ -function, which is bounded from below and satisfies  $(PS)_c^\phi$  on  $\mathcal{N}_{\Omega,V}^\phi$  for all  $c < \ell(\Gamma)c_\infty$ . Therefore, if  $d := \ell(\Gamma)(c_\infty - d_0e^{-\lambda R})$ , then  $J_V$  has at least

$$\text{genus}(\mathcal{N}_{\Omega,V}^\phi \cap J_V^d)$$

pairs of critical points  $\pm u$  with  $J_V(u) \leq d$ , where  $J_V^d := \{u \in H_0^1(\Omega) : J_V(u) \leq d\}$ . The map  $\theta : Z \rightarrow \mathcal{N}_{\Omega,V}^\phi \cap J_V^d$  defined by (4.1) is continuous. Furthermore,  $\theta(gz) = \theta(z)$  for all  $g \in G$  and  $\theta(\gamma z) = -\theta(z)$  if  $\phi(\gamma) = -1$ . Consequently,  $\theta$  induces a continuous map  $\hat{\theta} : Z/G \rightarrow \mathcal{N}_{\Omega,V}^\phi \cap J_V^d$ , given by  $\hat{\theta}(Gz) := \theta(z)$ , which satisfies  $\hat{\theta}((-1) \cdot Gz) = -\hat{\theta}(Gz)$  for all  $z \in Z$ . This implies that

$$\text{genus}(Z/G) \leq \text{genus}(\mathcal{N}_{\Omega,V}^\phi \cap J_V^d)$$

and concludes the proof.  $\square$

## 5. PROOF OF THEOREMS 1.3 AND 1.4

Let  $\phi : \Gamma \rightarrow \mathbb{Z}/2$  be a continuous group homomorphism and set  $G := \ker \phi$ . Let  $\omega \in H^1(\mathbb{R}^N)$  be a positive ground state of problem (2.7) which is radially symmetric about the origin, and let  $Z$  be a nonempty  $\Gamma$ -invariant subset of  $\Sigma$ . If  $\phi$  is an epimorphism, we also assume that  $Z \subset \Sigma \setminus \Sigma_0$ . Thus, for  $z \in Z$  and  $R > 0$ , the function

$$(5.1) \quad \sigma_{Rz} := \sum_{gz \in \Gamma z} \phi(g)\omega_{Rgz}, \quad \text{where } \omega_\zeta(x) := \omega(x - \zeta),$$

is well defined and  $\phi$ -equivariant (see Proposition 4.2). In addition, we assume that

$(Z_*) \quad \mu^\Gamma(Z) < 2$  and there exists  $a_0 > 1$  such that

$$\text{dist}(\gamma z, Gz) \geq a_0\mu(Gz) \quad \text{for any } z \in Z \text{ and } \gamma \in \Gamma \setminus G.$$

We choose  $R_0 > 0$  such that  $(\mathbb{R}^N \setminus \Omega) \subset B_{R_0}(0)$ , and a radially symmetric cut-off function  $\chi \in \mathcal{C}^\infty(\mathbb{R}^N)$  such that  $0 \leq \chi(x) \leq 1$ ,  $\chi(x) = 0$  if  $|x| \leq R_0$  and

$\chi(x) = 1$  if  $|x| \geq 2R_0$ . Observe that  $\chi\sigma_R \in H_0^1(\Omega)^\phi$ . We shall prove the following result.

**Proposition 5.1.** *If  $Z$  and  $V$  satisfy  $(Z_*)$  and  $(V_4)$  then there exist  $c_0, R_0 > 0$  and  $\beta > 1$  such that*

$$(5.2) \quad \frac{\|\chi\sigma_{Rz}\|_V^2}{\mathbb{D}(\chi\sigma_{Rz})^{\frac{1}{p}}} \leq (\ell(\Gamma) \|\omega\|^2)^{\frac{p-1}{p}} - c_0 e^{-\beta R} \quad \text{for any } R \geq R_0, z \in Z.$$

Consequently,  $c_{\Omega,V}^\phi < \ell(\Gamma)c_\infty$ .

We require some preliminary lemmas.

**Lemma 5.2.** (i) *If  $p \geq 2$  and  $a_1, \dots, a_n \geq 0$ , then*

$$\left| \sum_{i=1}^n a_i \right|^p \geq \sum_{i=1}^n a_i^p + (p-1) \sum_{i \neq k} a_i^{p-1} a_k.$$

(ii) *If  $p \geq 2$  and  $a, b \geq 0$ , then*

$$|a - b|^p \geq a^p + b^p - p(a^{p-1}b + ab^{p-1}).$$

*Proof.* See Lemma 4 in [4]. □

**Lemma 5.3.** *If  $p \geq 2$ ,  $A = \sum_{i=1}^n a_i$ ,  $\tilde{A} = \sum_{i=1}^n \tilde{a}_i$ ,  $B = \sum_{i=1}^n b_i$  and  $\tilde{B} = \sum_{i=1}^n \tilde{b}_i$  with  $a_i, \tilde{a}_i, b_i, \tilde{b}_i \geq 0$ , then*

$$(5.3) \quad A^p B^p \geq \sum_{i=1}^n a_i^p b_i^p + (p-1) \left( \sum_{j \neq m} a_j^p b_j^{p-1} b_m + \sum_{i \neq k} b_i^p a_i^{p-1} a_k \right),$$

$$(5.4) \quad A^2 B^2 \geq \sum_{i=1}^n a_i^2 b_i^2 + 2 \left( \sum_{j \neq m} a_j^2 b_j b_m + \sum_{i \neq k} b_i^2 a_i a_k \right),$$

$$(5.5) \quad \begin{aligned} |A - \tilde{A}|^p |B - \tilde{B}|^p &\geq A^p B^p + \tilde{A}^p \tilde{B}^p \\ &\quad - pn^{p-1} (B^p + \tilde{B}^p) \left[ \left( \sum_{i=1}^n a_i^{p-1} \right) \tilde{A} + \left( \sum_{i=1}^n \tilde{a}_i^{p-1} \right) A \right] \\ &\quad - pn^{p-1} (A^p + \tilde{A}^p) \left[ \left( \sum_{i=1}^n b_i^{p-1} \right) \tilde{B} + \left( \sum_{i=1}^n \tilde{b}_i^{p-1} \right) B \right]. \end{aligned}$$

*Proof.* Using Lemma 5.2(i) we obtain

$$\begin{aligned} \left| \sum_{i=1}^n a_i \right|^p \left| \sum_{j=1}^n b_j \right|^p &\geq \left( \sum_{i=1}^n a_i^p + (p-1) \sum_{i \neq k} a_i^{p-1} a_k \right) \left( \sum_{j=1}^n b_j^p + (p-1) \sum_{j \neq m} b_j^{p-1} b_m \right) \\ &\geq \sum_{i=1}^n a_i^p b_i^p + (p-1) \sum_{j \neq m} (a_j^p + a_m^p) b_j^{p-1} b_m + (p-1) \sum_{i \neq k} (b_i^p + b_k^p) a_i^{p-1} a_k. \end{aligned}$$

Inequalities (5.3) and (5.4) can be immediately deduced from the above expression. On the other hand, applying Lemma 5.2(ii) we obtain

$$\begin{aligned} &|A - \tilde{A}|^p |B - \tilde{B}|^p \\ &\geq [A^p + \tilde{A}^p - p(A^{p-1}\tilde{A} + A\tilde{A}^{p-1})] [B^p + \tilde{B}^p - p(B^{p-1}\tilde{B} + B\tilde{B}^{p-1})] \\ &\geq A^p B^p + \tilde{A}^p \tilde{B}^p - p(B^p + \tilde{B}^p)(A^{p-1}\tilde{A} + A\tilde{A}^{p-1}) - p(A^p + \tilde{A}^p)(B^{p-1}\tilde{B} + B\tilde{B}^{p-1}), \end{aligned}$$

which yields inequality (5.5).  $\square$

**Lemma 5.4.** *For every  $u \in H^1(\mathbb{R}^N)$  the following inequalities hold:*

$$\begin{aligned} \|\chi u\|_V^2 &\leq \|u\|_V^2 - \int_{\mathbb{R}^N} (\chi \Delta \chi) u^2, \\ \mathbb{D}(\chi u) &\geq \mathbb{D}(u) - 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(1 - \chi^p(x)) |u(x)|^p |u(y)|^p}{|x - y|^\alpha} dx dy. \end{aligned}$$

*Proof.* For every  $u \in H^1(\mathbb{R}^N)$  one has that

$$\begin{aligned} \|\chi u\|_V^2 &= \int_{\mathbb{R}^N} (|\chi \nabla u + u \nabla \chi|^2 + (1 + V(x)) |\chi u|^2) \\ &= \int_{\mathbb{R}^N} \chi^2 (|\nabla u|^2 + (1 + V(x)) |u|^2) + \int_{\mathbb{R}^N} (|\nabla \chi|^2 - \frac{1}{2} \Delta(\chi^2)) u^2 \\ &\leq \|u\|_V^2 - \int_{\mathbb{R}^N} (\chi \Delta \chi) u^2. \end{aligned}$$

Writing  $ab = 1 - (1 - a) - (1 - b) + (1 - a)(1 - b)$  and taking  $a := \chi^p(x)$ ,  $b := \chi^p(y)$ , we obtain

$$\begin{aligned} \mathbb{D}(\chi u) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi^p(x) \chi^p(y) |u(x)|^p |u(y)|^p}{|x - y|^\alpha} dx dy \\ &= \mathbb{D}(u) - 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(1 - \chi^p(x)) |u(x)|^p |u(y)|^p}{|x - y|^\alpha} dx dy \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(1 - \chi^p(x))(1 - \chi^p(y)) |u(x)|^p |u(y)|^p}{|x - y|^\alpha} dx dy. \end{aligned}$$

Notice that the last summand in the right-hand side of the above expression is nonnegative. Then the second inequality follows.  $\square$

We shall apply this lemma to the function  $\sigma_{Rz}$  to derive inequality (5.2). To this purpose we also require some asymptotic estimates, which will be provided by the following four lemmas.

Since  $\omega$  is a solution of problem (2.7), for any  $z, z' \in \mathbb{R}^N$ , one has that  $J'_\infty(\omega_z) \omega_{z'} = 0$ , which is equivalent to

$$\int_{\mathbb{R}^N} [\nabla \omega_z \cdot \nabla \omega_{z'} + \omega_z \omega_{z'}] = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} * \omega_z^p \right) \omega_z^{p-1} \omega_{z'}.$$

A change of variable in the right-hand side of this inequality allows us to express it as

$$(5.6) \quad \langle \omega_z, \omega_{z'} \rangle = I(z' - z) \quad \text{for all } z, z' \in \mathbb{R}^N,$$

where  $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $H^1(\mathbb{R}^N)$  and  $I$  is the function defined in (3.2). We denote by  $Fz := \{(gz, hz) \in \Gamma z \times \Gamma z : gz \neq hz\}$  and define

$$\begin{aligned} \varepsilon_{Rz} &:= \sum_{\substack{(gz, hz) \in Fz \\ \phi(g) = \phi(h)}} I(Rgz - Rhz), \\ \widehat{\varepsilon}_{Rz} &:= \sum_{\substack{(gz, hz) \in Fz \\ \phi(g) \neq \phi(h)}} I(Rgz - Rhz) \text{ if } \phi \not\equiv 1, \quad \text{and} \quad \widehat{\varepsilon}_{Rz} := 0 \text{ if } \phi \equiv 1. \end{aligned}$$

We choose  $g_z, h_z \in Gz$  such that  $|g_z z - h_z z| = \mu(\Gamma z) := \min\{|gz - hz| : g, h \in \Gamma, gz \neq hz\}$  and set

$$\xi_z := g_z z - h_z z.$$

**Lemma 5.5.** *If  $(Z_*)$  holds, then*

$$\widehat{\varepsilon}_{Rz} = o(\varepsilon_{Rz})$$

*uniformly in  $z \in Z$ .*

*Proof.* For  $a_0 > 1$  as in condition  $(Z_*)$  we fix  $\widehat{a} \in (0, 1)$  such that  $a := \widehat{a}a_0 > 1$ . Thus,  $a|\xi_z| = a\mu(Gz) \leq \widehat{a}|gz - hz|$  for any  $z \in Z$ ,  $g, h \in \Gamma$  with  $gz \neq hz$  and  $\phi(g) \neq \phi(h)$ . Lemma 3.5 yields a constant  $k_a > 0$  such that

$$I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|} \geq k_a \quad \text{if } R \geq \mu_\Gamma(Z)^{-1},$$

where  $b := \frac{N-1}{2}$ . So, setting  $C := k_a^{-1}$  we obtain

$$\begin{aligned} \frac{I(Rgz - Rhz)}{I(R\xi_z)} &\leq \frac{I(Rgz - Rhz)|Rgz - Rhz|^b e^{\widehat{a}|Rgz - Rhz|}}{I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|}} \\ &\leq CI(Rgz - Rhz)|Rgz - Rhz|^b e^{\widehat{a}|Rgz - Rhz|} \quad \text{if } R \geq \mu_\Gamma(Z)^{-1}. \end{aligned}$$

Let  $\varepsilon > 0$ . Lemma 3.4 asserts that there exists  $S > 0$  such that  $I(\zeta)|\zeta|^b e^{\widehat{a}|\zeta|} < \varepsilon$  if  $|\zeta| > S$ . As  $\widehat{a}|Rgz - Rhz| \geq Ra\mu_G > 0$ , taking  $R_0 := \max\{\frac{\widehat{a}S}{a\mu_G}, \mu_\Gamma(Z)^{-1}\}$  we conclude that

$$0 \leq \frac{\widehat{\varepsilon}_{Rz}}{\varepsilon_{Rz}} \leq \sum_{\substack{gz \neq hz \in \Gamma z \\ \phi(g) \neq \phi(h)}} \frac{I(Rgz - Rhz)}{I(R\xi_z)} \leq \ell(G)^2 C\varepsilon \quad \text{if } R \geq R_0,$$

which proves the assertion.  $\square$

**Lemma 5.6.** *If  $(Z_*)$  holds then, for any  $g, h \in \Gamma$  such that  $\phi(g) \neq \phi(h)$  and  $\gamma \in \Gamma \setminus G$ , we have that*

$$\int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} * \left( \left| \sum_{\zeta \in Gz} \omega_{R\zeta} \right|^p + \left| \sum_{\zeta \in Gz} \omega_{R\gamma\zeta} \right|^p \right) \right) \omega_{Rgz}^{p-1} \omega_{Rh\zeta} = o(\varepsilon_{Rz})$$

*uniformly in  $z \in Z$ .*

*Proof.* Since  $\frac{1}{|x|^\alpha} * \omega^p \in L^\infty(\mathbb{R}^N)$ , we have that  $\frac{1}{|x|^\alpha} * \left( \left| \sum_{\zeta \in Gz} \omega_{R\zeta} \right|^p + \left| \sum_{\zeta \in Gz} \omega_{R\gamma\zeta} \right|^p \right)$  is bounded on  $\mathbb{R}^N$  uniformly in  $z$ . Hence,

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} * \left( \left| \sum_{\zeta \in Gz} \omega_{R\zeta} \right|^p + \left| \sum_{\zeta \in Gz} \omega_{R\gamma\zeta} \right|^p \right) \right) \omega_{Rgz}^{p-1} \omega_{Rh\zeta} \\ &\leq C \int_{\mathbb{R}^N} \omega_{Rgz}^{p-1} \omega_{Rh\zeta} = C \int_{\mathbb{R}^N} \omega^{p-1} \omega_{R(hz-gz)}. \end{aligned}$$

Arguing as in Lemma 5.5, using this time Lemma 3.3, we obtain the conclusion.  $\square$

**Lemma 5.7.** *If  $Z$  and  $V$  satisfy  $(Z_*)$  and  $(V_4)$ , then*

$$\int_{\mathbb{R}^N} V^+ \sigma_{Rz}^2 = o(\varepsilon_{Rz})$$

*uniformly in  $z \in Z$ .*

*Proof.* Let  $\kappa > \mu^\Gamma(Z)$  be as in assumption  $(V_4)$  (recall that  $V_\infty = 1$  is assumed). We fix  $a > 1$  such that  $M := a\mu^\Gamma(Z) < \min\{2, \kappa\}$ . Lemma 3.5 implies that there exists a positive constant  $k_a$  such that

$$I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|} \geq k_a \quad \text{if } R \geq \mu_\Gamma(Z)^{-1},$$

where  $b := \frac{N-1}{2}$ . Observing that  $M|Rz| = MR = aR\mu^\Gamma(Z) \geq a|R\xi_z|$  for all  $z \in Z$ , we conclude that

$$\begin{aligned} \frac{\int_{\mathbb{R}^N} V^+ \sigma_{Rz}^2}{\varepsilon_{Rz}} &\leq C \sum_{gz \in \Gamma z} \frac{A(Rgz)}{I(R\xi_z)} \leq C \sum_{gz \in \Gamma z} \frac{A(Rgz)|Rgz|^b e^{M|Rgz|}}{I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|}} \\ &\leq C \sum_{gz \in \Gamma z} A(Rgz)|Rgz|^b e^{M|Rgz|} \quad \text{if } R \geq \mu_\Gamma(Z)^{-1}, \end{aligned}$$

where  $C$  denotes different positive constants and  $A$  is the map defined in (3.3). Taking Lemma 3.7 into account, we obtain that

$$\lim_{R \rightarrow \infty} \frac{\int_{\mathbb{R}^N} V^+ \sigma_{Rz}^2}{\varepsilon_{Rz}} = 0$$

uniformly in  $z \in Z$ , as claimed.  $\square$

**Lemma 5.8.** *If  $f \in C_c^0(\mathbb{R}^N)$  and  $q > \max\{\mu^\Gamma(Z), 1\}$ , then*

$$\int_{\mathbb{R}^N} f \sigma_{Rz}^q = o(\varepsilon_{Rz})$$

*uniformly in  $z \in Z$ .*

*Proof.* Let us fix  $a > 1$  such that  $\hat{a} := \frac{a\mu^\Gamma(Z)}{q} < 1$ . Lemma 3.5 yields that there exists  $k_a > 0$  such that

$$I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|} \geq k_a \quad \text{if } R \geq \mu_\Gamma(Z)^{-1},$$

where  $b := \frac{N-1}{2}$ . Since  $q\hat{a}|Rz| = q\hat{a}R = aR\mu^\Gamma(Z) \geq a|R\xi_z|$  for all  $z \in Z$ , we conclude that

$$\begin{aligned} \frac{\int_{\mathbb{R}^N} |f| \sigma_{Rz}^q}{\varepsilon_{Rz}} &\leq C \sum_{gz \in \Gamma z} \frac{\int_{\mathbb{R}^N} |f| \omega_{Rgz}^q}{I(R\xi_z)} \leq C \sum_{gz \in \Gamma z} \frac{\int_{\mathbb{R}^N} |f| \omega_{Rgz}^q |Rgz|^b e^{q\hat{a}|Rgz|}}{I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|}} \\ &\leq C \sum_{gz \in \Gamma z} \int_{\mathbb{R}^N} |f| \omega_{Rgz}^q |Rgz|^b e^{q\hat{a}|Rgz|} \quad \text{if } R \geq \mu_\Gamma(Z)^{-1}, \end{aligned}$$

where  $C$  denote distinct positive constants. Hence, from Lemma 3.8 we get

$$\lim_{R \rightarrow \infty} \frac{\int_{\mathbb{R}^N} f \sigma_{Rz}^q}{\varepsilon_{Rz}} = 0$$

uniformly in  $z \in Z$ .  $\square$

Finally, we need the following result.

**Lemma 5.9.** *Let  $\psi : (0, \infty) \rightarrow \mathbb{R}$  be the function given by*

$$\psi(t) := \frac{a + t + o(t)}{(a + bt + o(t))^\beta},$$

*where  $a > 0$ ,  $\beta \in (0, 1)$  and  $b\beta > 1$ . Then, there exist constants  $c_0, t_0 > 0$  such that*

$$\psi(t) \leq a^{1-\beta} - c_0 t \quad \text{for all } t \in (0, t_0).$$

*Proof.* Taking  $\frac{1}{\beta} < q < b$  and  $1 < s < r < \beta q$ , we have that there exists  $t_1 \in (0, 1)$  such that

$$\psi(t) \leq \frac{a + st}{(a + qt)^\beta} = \frac{a + rt}{(a + qt)^\beta} - \frac{(r - s)t}{(a + qt)^\beta} \quad \text{for all } t \in (0, t_1).$$

We denote by  $f(t) := \frac{a + rt}{(a + qt)^\beta}$ . Since  $f'(0) = (r - \beta q) a^{-\beta} < 0$ , there exists  $t_0 \in (0, t_1)$  such that

$$f(t) \leq f(0) = a^{1-\beta} \quad \text{for all } t \in (0, t_0).$$

Consequently,

$$\psi(t) \leq a^{1-\beta} - \frac{(r - s)}{(a + q)^\beta} t \quad \text{for all } t \in (0, t_0),$$

which concludes the proof.  $\square$

*Proof of Proposition 5.1.* Let  $\gamma \in \Gamma \setminus G$ . If  $Gz = \{z_1, \dots, z_\ell\}$  with  $\ell := \ell(G)$ , we write

$$\sigma_{Rz} = \sigma_{Rz}^1 - \sigma_{Rz}^2 \quad \text{with} \quad \sigma_{Rz}^1 := \sum_{i=1}^{\ell} \omega_{Rz_i} \quad \text{and} \quad \sigma_{Rz}^2 := \sum_{i=1}^{\ell} \omega_{R\gamma z_i}.$$

Applying Lemma 5.3 to  $a_i := \omega_{Rz_i}(x)$ ,  $\hat{a}_i := \omega_{R\gamma z_i}(x)$ ,  $b_i := \omega_{Rz_i}(y)$ ,  $\hat{b}_i := \omega_{R\gamma z_i}(y)$  and using Lemma 5.6 we conclude that

$$\begin{aligned} \mathbb{D}(\sigma_{Rz}) &\geq \mathbb{D}(\sigma_{Rz}^1) + \mathbb{D}(\sigma_{Rz}^2) + o(\varepsilon_{Rz}) \\ &\geq \begin{cases} \ell(\Gamma)\mathbb{D}(\omega) + 2(p-1)\varepsilon_{Rz} + o(\varepsilon_{Rz}) & \text{if } p > 2, \\ \ell(\Gamma)\mathbb{D}(\omega) + 4\varepsilon_{Rz} + o(\varepsilon_{Rz}) & \text{if } p = 2. \end{cases} \end{aligned}$$

Note that, since  $\frac{1}{|x|^\alpha} * \omega^p \in L^\infty(\mathbb{R}^N)$ ,  $\frac{1}{|x|^\alpha} * |\sigma_{Rz}|^p$  is bounded uniformly in  $z$ . So, since  $\mu^\Gamma(Z) < 2 \leq p$ ,  $\chi \Delta \chi \in C_c^0(\mathbb{R}^N)$  and  $1 - \chi^p \in C_c^0(\mathbb{R}^N)$ , Lemma 5.8 yields that

$$\int_{\mathbb{R}^N} (\chi \Delta \chi) \sigma_{Rz}^2 = o(\varepsilon_{Rz}) \quad \text{and} \quad \int_{\mathbb{R}^N} (1 - \chi^p) \left( \frac{1}{|x|^\alpha} * |\sigma_{Rz}|^p \right) \sigma_{Rz}^p = o(\varepsilon_{Rz})$$

uniformly in  $z$ . This, together with Lemmas 5.4, 5.5 and 5.7 and expression (5.6), yields

$$\begin{aligned} \|\chi \sigma_{Rz}\|_V^2 &\leq \|\sigma_{Rz}\|^2 + \int_{\mathbb{R}^N} V \sigma_{Rz}^2 - \int_{\mathbb{R}^N} (\chi \Delta \chi) \sigma_{Rz}^2 \\ &\leq \ell(\Gamma) \|\omega\|^2 + \varepsilon_{Rz} - \widehat{\varepsilon}_{Rz} + \int_{\mathbb{R}^N} V^+ \sigma_{Rz}^2 + o(\varepsilon_{Rz}) \\ &\leq \ell(\Gamma) \|\omega\|^2 + \varepsilon_{Rz} + o(\varepsilon_{Rz}), \\ \mathbb{D}(\chi \sigma_{Rz}) &\geq \ell(\Gamma)\mathbb{D}(\omega) + b_p \varepsilon_{Rz} + o(\varepsilon_{Rz}) - 2 \int_{\mathbb{R}^N} (1 - \chi^p) \left( \frac{1}{|x|^\alpha} * |\sigma_{Rz}|^p \right) \sigma_{Rz}^p \\ &\geq \ell(\Gamma)\mathbb{D}(\omega) + b_p \varepsilon_{Rz} + o(\varepsilon_{Rz}), \end{aligned}$$

where  $b_p := 2(p-1)$  if  $p > 2$  and  $b_p := 4$  if  $p = 2$ . Consequently, since  $\|\omega\|^2 = \mathbb{D}(\omega)$  and  $\varepsilon_{Rz} \rightarrow 0$  as  $R \rightarrow \infty$  uniformly in  $z$ , Lemma 5.9 insures that there exist  $c_1, R_1 > 0$  such that

$$\frac{\|\chi \sigma_{Rz}\|_V^2}{(\mathbb{D}(\chi \sigma_{Rz}))^{\frac{1}{p}}} \leq \frac{\ell(\Gamma) \|\omega\|^2 + \varepsilon_{Rz} + o(\varepsilon_{Rz})}{(\ell(\Gamma)\mathbb{D}(\omega) + b_p \varepsilon_{Rz} + o(\varepsilon_{Rz}))^{\frac{1}{p}}} \leq (\ell(\Gamma) \|\omega\|^2)^{\frac{p-1}{p}} - c_1 \varepsilon_{Rz}$$

for  $R \geq R_1$  and  $z \in Z$ . Using Lemma 3.5 we conclude that there exist  $c_0, R_0 > 0$  and  $\beta > 1$  such that

$$\frac{\|\chi\sigma_{Rz}\|_V^2}{\mathbb{D}(\chi\sigma_{Rz})^{\frac{1}{p}}} \leq (\ell(\Gamma) \|\omega\|^2)^{\frac{p-1}{p}} - c_0 e^{-\beta R} \quad \text{for any } R \geq R_0, z \in Z,$$

which is inequality (5.2). Finally, since  $\pi(\chi\sigma_{Rz}) \in \mathcal{N}_{\Omega,V}^\phi$  and

$$J_V(\pi(\chi\sigma_{Rz})) = \frac{p-1}{2p} \left( \frac{\|\chi\sigma_{Rz}\|_V^2}{\mathbb{D}(\chi\sigma_{Rz})^{\frac{1}{p}}} \right)^{\frac{p}{p-1}} < \frac{p-1}{2p} \ell(\Gamma) \|\omega\|^2 = \ell(\Gamma) c_\infty,$$

one has that  $c_{\Omega,V}^\phi < \ell(\Gamma) c_\infty$ .  $\square$

*Proof of Theorem 1.3.* Let  $\phi \equiv 1$ , so that  $\Gamma = G$ . If assumption  $(V_3)$  holds for  $\kappa > \mu_G$ , we choose  $\zeta \in \Sigma$  such that  $\mu(G\zeta) \in [\mu_G, \kappa)$  and set  $Z := G\zeta$ . Thus  $\mu^G(Z) = \mu(G\zeta)$  and assumption  $(V_4)$  holds for  $\kappa$ . Moreover, since  $\ell(G) \geq 3$ ,  $\mu^G(Z) = \mu(G\zeta) < 2$ . Therefore  $(Z_*)$  holds and we can apply Proposition 5.1 to these data to conclude that  $c_{\Omega,V}^G < \ell(G) c_\infty$ . Corollary 2.3 then insures that  $J_V$  satisfies condition  $(PS)_c^G$  on  $\mathcal{N}_{\Omega,V}^G$  for  $c := c_{\Omega,V}^G$ . Consequently, there exists  $u \in \mathcal{N}_{\Omega,V}^G$  such that  $J_V(u) = c_{\Omega,V}^G$ . Since  $|u| \in \mathcal{N}_{\Omega,V}^G$  and  $J_V(|u|) = J_V(u)$ ,  $|u|$  is a positive solution of (1.1) which is  $G$ -invariant and satisfies  $J_V(|u|) < \ell(G) c_\infty$ .  $\square$

*Proof of Theorem 1.4.* If  $\phi$  is an epimorphism and  $(Z_0)$  holds, then  $Z \subset \Sigma \setminus \Sigma_0$  and  $2 > \frac{2}{a_0} \geq \mu(Gz) = \mu(\Gamma z)$ . Therefore,  $\mu^\Gamma(Z) < 2$ , and hence  $(Z_*)$  holds. We choose  $R > R_0$  and set  $d := \frac{p-1}{2p} [(\ell(\Gamma) \|\omega\|^2)^{\frac{p-1}{p}} - c_0 \varepsilon^{-\beta R}]^{\frac{p}{p-1}}$ . Proposition 5.1 then asserts that the map  $\sigma : Z \rightarrow \mathcal{N}_{\Omega,V}^\phi \cap J_V^d$  given by  $\sigma(z) := \pi(\chi\sigma_{Rz})$  is well defined. Furthermore,  $\sigma(gz) = \sigma(z)$  for all  $g \in G$  and  $\sigma(\gamma z) = -\sigma(z)$  if  $\phi(\gamma) = -1$ . Consequently,  $\sigma$  induces a continuous map  $\widehat{\sigma} : Z/G \rightarrow \mathcal{N}_{\Omega,V}^\phi \cap J_V^d$ , given by  $\widehat{\sigma}(Gz) := \sigma(z)$ , which satisfies  $\widehat{\sigma}((-1) \cdot Gz) = -\widehat{\sigma}(Gz)$  for all  $z \in Z$ . This implies that

$$\text{genus}(Z/G) \leq \text{genus}(\mathcal{N}_{\Omega,V}^\phi \cap J_V^d).$$

Since  $\mathcal{N}_{\Omega,V}^\phi$  is a  $\mathcal{C}^2$ -manifold and  $J_V : \mathcal{N}_{\Omega,V}^\phi \rightarrow \mathbb{R}$  is an even  $\mathcal{C}^2$ -function which is bounded from below and satisfies condition  $(PS)_c^\phi$  on  $\mathcal{N}_{\Omega,V}^\phi$  for all  $c < \ell(\Gamma) c_\infty$ , we conclude that  $J_V$  has at least  $\text{genus}(Z/G)$  pairs of critical points  $\pm u$  with  $J_V(u) \leq d$ .  $\square$

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INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CIRCUITO EXTERIOR, C.U., 04510 MÉXICO D.F., MEXICO.

*E-mail address:* Mónica Clapp <mclapp@matem.unam.mx>

*E-mail address:* Dora Salazar <docesalo@gmail.com>